

# ON THE CONJUGACY PROBLEM IN GROUP $F/N_1 \cap N_2$ .

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АННОТАЦИЯ. Let  $N_1$  (resp.,  $N_2$ ) be the normal closure of a finite symmetrized set  $R_1$  (resp.,  $R_2$ ) of a finitely generated free group  $F = F(A)$ . It is well-known that if  $R_i$  satisfies the condition  $C(6)$ , then the conjugacy problem is solvable in  $F/N_i$ . In the present paper we prove that if  $R_1 \cup R_2$  satisfies the condition  $C(6)$  and the presentation  $\langle A \mid R_1, R_2 \rangle$  is atorical, then the conjugacy problem is solvable in  $F/N_1 \cap N_2$ . In particular, if  $R_1 \cup R_2$  satisfies the condition  $C(7)$  then the conjugacy problem is solvable in  $F/N_1 \cap N_2$ .

Bibliography: 13 items.

## INTRODUCTION.

Let  $F = F(A)$  be a free group generated by a finite alphabet  $A$ . Let  $N_1$  (resp.,  $N_2$ ) be the normal closure of non-empty finite set  $R_1$  (resp.,  $R_2$ ) of elements of  $F$ . Assume that  $R_i$  ( $i = 1, 2$ ) is symmetrized, i.e., all elements of  $R_i$  are cyclically reduced and for any  $r$  of  $R_i$  all cyclic permutations of  $r$  and  $r^{-1}$  also belong to  $R_i$ .

We will use the following notations. Denote graphic (letter-by-letter) equality of words  $u, v \in F$  by  $u \equiv v$ . Denote free equality by  $u = v$ . If words  $u, v \in F$  present equal elements in a group  $H$ , we will write:  $u = v$  in  $H$ .

If two words  $u, v \in F$  are equal both in the group  $F/N_1$  and in the group  $F/N_2$ , then they are evidently equal in the group  $F/N_1 \cap N_2$ . It is natural to ask whether the conjugation of words  $u$  and  $v$  in  $F/N_1 \cap N_2$  follows from their conjugation both in  $F/N_1$  and  $F/N_2$ ? The answer is obviously negative. As an example showing that one can consider the free group  $F = F(a, b, c)$ , the sets  $R_1 = \{a^{\pm 1}\}$ ,  $R_2 = \{b^{\pm 1}\}$  and the words  $u \equiv c^2ba$ ,  $v \equiv cbca$ .

The aim of this paper is to find out conditions on  $R_1$  and  $R_2$  such that the solvability of the conjugacy problem in  $F/N_1 \cap N_2$  follows from the solvability of the conjugacy problem in  $F/N_1$  and  $F/N_2$ .

Note that this problem is naturally associated with subdirect products. Indeed, one can consider  $F/N_1 \cap N_2$  as a subgroup of the direct product of  $F/N_1$  and  $F/N_2$ , and  $F/N_1 \cap N_2$  is a subdirect product of  $F/N_1$  and

$F/N_2$ . Conversely, given a subdirect product  $H$  of groups  $G_1$  and  $G_2$ , there exist normal subgroups  $N_1$  and  $N_2$  of some free group  $F$  such that  $F/N_i \cong G_i$  ( $i=1,2$ ) and  $F/N_1 \cap N_2 \cong H$ .

In turn, subdirect products of two groups are closely associated to the fibre product construction in the category of groups. Recall that, associated to each pair of short exact sequences of groups  $1 \rightarrow L_i \rightarrow G_i \xrightarrow{\Psi_i} Q \rightarrow 1$ ,  $i = 1, 2$ , one has the *fibre product*  $H = \{(x, y) \in G_1 \times G_2 \mid \Psi_1(x) = \Psi_2(y)\}$ . It is shown in [1] that a subgroup  $H \leq G_1 \times G_2$  is a subdirect product of  $G_1$  and  $G_2$  if and only if there is a group  $Q$  and surjections  $\Psi_i : G_i \rightarrow Q$  such that  $H$  is the fibre product of  $\Psi_1$  and  $\Psi_2$ .

The question about the solvability of the conjugacy problem for subdirect products has been already considered for some groups (see, for example, [1, 2, 3]). Thus in the paper of C.F. Miller [2] there is an example of a fibre product in which  $G_1 = G_2$  are non-abelian finitely generated free groups,  $L_1 = L_2$ ,  $\Psi_1 = \Psi_2$ ,  $Q$  is a finitely presented group with undecidable word problem, and the conjugacy problem in  $H$  is unsolvable. So the natural question, whether the solvability of the conjugacy problem in  $F/N_1 \cap N_2$  always follows from the solvability of the conjugacy problem both in  $F/N_1$  and in  $F/N_2$ , has negative answer. Since  $Q$  is isomorphic to  $F/N_1 N_2$ , it follows from the example of C.F. Miller that the solvability of the word problem in  $F/N_1 N_2$  is necessary for the solvability of the conjugacy problem in  $F/N_1 \cap N_2$ .

To formulate the main result of the paper, we recall the definitions of some geometrical objects, called pictures. Pictures were introduced in [4, 5]. These objects are a very useful tool in combinatorial group theory, and can be used in a variety of different ways (see, for example, [6, 7] and references in these papers).

Let  $N$  be the normal closure of a symmetrized set  $R$  of the free group  $F(A)$ .

A *picture*  $P$  over the presentation  $\hat{G} = \langle A \mid R \rangle$  on an oriented surface  $T$  is a finite collection of "vertices"  $V_1, \dots, V_n \in T$ , together with a finite collection of simple pairwise disjoint connected oriented "edges"  $E_1, \dots, E_m \in T \setminus \{\{V_1, \dots, V_n\} \cup \partial T\}$  labelled by words of  $F(A)$ . But these edges need not all connect two vertices. An edge may connect a vertex to a vertex (possibly coincident), a vertex to  $\partial T$ , or  $\partial T$  to  $\partial T$ . Moreover, some edges need have no endpoints at all, but be circles disjoint from the rest of  $P$ , such edges are called edges-circles.

In the paper we will only ever consider such paths on  $T$ , each of which doesn't pass through any vertex and intersects the edges of  $P$  only finitely many times (moreover, if a path intersects an edge then

it crosses it, and doesn't just touch it). If we travel along an oriented path  $\gamma$  in the positive direction, we encounter a succession of edges  $E_{i_1}, \dots, E_{i_k}$  labeled by  $g_{i_1}, \dots, g_{i_k}$  respectively. These labels form the word  $g_{i_1}^{\varepsilon_{i_1}} \cdot \dots \cdot g_{i_k}^{\varepsilon_{i_k}}$ , where  $\varepsilon_{i_j} \in \{1, -1\}$  is a local intersection index of  $E_{i_j}$  and  $\gamma$ . This word will be called *the word along the path  $\gamma$*  (or *the label of  $\gamma$* ) and denoted by  $Lab^+(\gamma)$ . The subword  $g_{i_j}^{\varepsilon_{i_j}}$  ( $j = 1, \dots, k$ ) will be called *the contribution of  $E_{i_j}$  in the label of  $\gamma$* . Travelling along  $\gamma$  in the negative direction gives the word  $Lab^-(\gamma) \equiv Lab^+(\gamma)^{-1}$ .

If a path  $\gamma$  is closed, consider a point  $p$  on  $\gamma$  not belonging to any edge of  $P$ . The word along  $\gamma$  read from  $p$  will be denoted by  $Lab_p^+(\gamma)$  or by  $Lab_p^-(\gamma)$  (depending on the direction of travelling along  $\gamma$ ). Changing the disposition of  $p$  we obtain the same word up to cyclic permutation. We will denote the word along the path  $\gamma$  by  $Lab(\gamma)$  when the disposition of  $p$  and the direction of reading will not be essential.

For each vertex  $V$  of  $P$  consider a circle  $\Sigma$  of a small radius with center at  $V$  and a point  $p \in \Sigma$  not lying on any edge of  $P$ . The word  $Lab_p^+(\Sigma)$  is called *the label of the vertex  $V$* . To complete the definition of the picture over the presentation  $\hat{G} = \langle A \mid R \rangle$  on the surface  $T$  it remains to require that the labels of all vertices in  $P$  belong to  $R$ .

Below we will consider pictures on a surface  $T$ , where  $T$  is a torus (torical pictures), an annulus (annulus pictures) or a disk (planar pictures).

For a planar picture *the boundary label* of the picture is the word  $Lab_{\bar{p}}^+(\bar{\Sigma})$ , where  $\bar{\Sigma}$  is a circle near the boundary of the disk  $T$  and  $\bar{p} \in \bar{\Sigma}$  is a point not belonging to any edge.

The following result is well-known (use Theorem 11.1 [9] and dualise):

**Lemma 1.** *Let  $W$  be a non-empty word on the alphabet  $A$ . Then  $W$  represents the identity of the group  $\hat{G} = F/N$  if and only if there is a planar picture over the presentation  $\langle A \mid R \rangle$  of  $\hat{G}$  with the boundary label  $W$ .*

A *dipole* is two distinct vertices  $V_1$  and  $V_2$  of  $P$  connected by an edge  $E$  if there exists a simple path  $\psi$  joining points  $p_1$  and  $p_2$  on the circles  $\Sigma_1$  and  $\Sigma_2$  around these vertices, passing along  $E$  and not crossing any edge or vertex such that  $Lab_{p_1}^+(\Sigma_1) = Lab_{p_2}^-(\Sigma_2)$  in  $F$ .

A presentation  $\hat{G} = \langle A \mid R \rangle$  is called *atorical* (see, for example, [9]) if every connected torical picture over  $\hat{G} = \langle A \mid R \rangle$  having at least one vertex contains a dipole.

The following theorem 1 will be proved in Section 1.

**Theorem 1.** *Let  $F$  be a free group generated by a finite alphabet  $A$ ,  $N_1$  (resp.,  $N_2$ ) be the normal closure of non-empty finite symmetrized set  $R_1$  (resp.,  $R_2$ ) of elements of  $F$ .*

*Let the following conditions hold for the group  $G_i = F/N_i$  ( $i = 1, 2$ ):*

*1.1. The conjugacy problem is solvable in  $G_i$ .*

*1.2. In  $G_i$ , there exists an algorithm allowing for a reduced word  $x \in F$ ,  $x \neq 1$ , to determine all  $z \in F$  such that  $x \in \langle z \rangle$  in  $G_i$ , and the number of such distinct elements  $z$  of  $G_i$  is finite.*

*Let the following conditions hold for the group  $G = F/N_1N_2$ :*

*2.1. The membership problem for a cyclic subgroup is solvable in  $G$ .*

*2.2. The presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$  is atorical.*

*Then the conjugacy problem is solvable in  $F/N_1 \cap N_2$ .*

Note that Condition 2.2 of Theorem 1 provides the equality  $N_1 \cap N_2 = [N_1, N_2]$  for disjoint  $R_1$  and  $R_2$  (see, for example, [10, 11]).

Recall the definition of small cancelation conditions  $C(k)$  ([8]), used in Theorem 2 below. A nontrivial freely reduced word  $b$  in  $F$  is called a *piece* with respect to  $R$  if there exist two distinct elements  $r_1$  and  $r_2$  in  $R$  that both have  $b$  as maximal initial segment, i.e.  $r_1 \equiv bc_1$  and  $r_2 \equiv bc_2$ . Let  $k$  be a positive integer.  $R$  is said to satisfy *the small cancelation condition  $C(k)$* , if no element of  $R$  can be written as a reduced product of fewer than  $k$  pieces.

Using the notations of Theorem 1, we have:

**Theorem 2.** *If  $R_1 \cup R_2$  is a set satisfying the condition  $C(6)$  and the presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$  is atorical, then the conjugacy problem is solvable in  $F/N_1 \cap N_2$ .*

*Proof of Theorem 2.* Let us show that Theorem 2 follows from Theorem 1. Since  $R_1 \cup R_2$  satisfies the condition  $C(6)$ , the subsets  $R_1$  and  $R_2$  also satisfy the condition  $C(6)$ . Therefore Condition 1.1 for  $G_1$  and  $G_2$  follows from Theorem 7.6 [8]; Condition 2.1 follows from Theorem 1 [12]; Condition 1.2 can be deduced from Theorem 1 [12], Theorem 2 [12] and (if there is an element of finite order in  $G_i$ ) Theorem 1.4 [13] with regard to Theorem 13.3 [9]. ■

It is well known that the condition  $C(7)$  is sufficient for atoricity (the proof of it is similar to Theorem 13.3 [9]). So by Theorem 2 (using the notations of Theorem 1) we have the following:

**Corollary 1.** *If  $R_1 \cup R_2$  satisfies the condition  $C(7)$ , then the conjugacy problem is solvable in  $F/N_1 \cap N_2$ .*

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# 1. DEDUCTION OF THEOREM 1 FROM ASSERTION 1.

Below, no mentioning it explicitly, we will use the fact that Condition 1.1 (resp., 2.1) of Theorem 1 leads to the solvability of the word problem in  $G_i$ ,  $i \in \{1, 2\}$  (resp.,  $G$ ).

Let  $u$  and  $v$  be two reduced words of  $F$ . For each  $i \in \{1, 2\}$  by Condition 1.1 of Theorem 1 there exists an algorithm which decides whether  $u$  and  $v$  present conjugated elements in  $G_i = F/N_i$ . If  $u$  and  $v$  turn out to be not conjugated in  $G_i$  for at least one of the  $i$ , then  $u$  and  $v$  are not conjugated in  $F/N_1 \cap N_2$ . Hence further assume that for each  $i \in \{1, 2\}$ ,  $u$  and  $v$  are conjugated by  $h_i \in F$  in  $G_i$ . Therefore the word  $h_i^{-1}uh_iv^{-1}$  is equal to the identity in  $G_i$ .

By Condition 1.1 of Theorem 1 the word  $h_1^{-1}uh_1v^{-1}$  can be effectively represented with defining relations  $R_1$  of  $G_1$  in the form  $\prod_{s=1}^{m_1} g_{1,s}r_{1,s}g_{1,s}^{-1}$ , where  $r_{1,s} \in R_1$ ,  $g_{1,s} \in F$ . By this representation construct a planar picture  $P_1$  over the presentation  $G_1 = \langle A \mid R_1 \rangle$  with the boundary label equal to  $h_1^{-1}uh_1v^{-1}$  so that the edges of  $P_1$  are labelled by letters. In addition on the boundary  $\partial P_1$  of  $P_1$  fix four points  $a_1, b_1, c_1, d_1$  not belonging to any edge and dividing  $\partial P_1$  into four subpaths so that the labels of the subpaths  $[a_1, b_1]$ ,  $[b_1, c_1]$ ,  $[c_1, d_1]$ ,  $[d_1, a_1]$  are identically equal to  $h_1^{-1}$ ,  $u$ ,  $h_1$ ,  $v^{-1}$  respectively. Pasting together the subpaths  $[a_1, b_1]$  and  $[d_1, c_1]$  of  $P_1$ , we obtain an annulus picture  $\overline{P}_1$  with the two boundary circles formed by  $[b_1, c_1]$  with the label  $u$  and  $[d_1, a_1]$  with the label  $v^{-1}$ . The pasted points  $b_1, c_1$  (resp.,  $d_1, a_1$ ) give a point  $(bc)_1$  (resp.,  $(ad)_1$ ). The pasted subpaths  $[a_1, b_1]$  and  $[d_1, c_1]$  form a subpath  $Conj_1$ .

Similarly changing the index 1 by the index 2 in the notation, by the word  $h_2^{-1}u^{-1}h_2v$  construct an annulus picture  $\overline{P}_2$  over the presentation  $G_2 = \langle A \mid R_2 \rangle$  with the two boundary circles formed by  $[b_2, c_2]$  with the label  $u^{-1}$  and  $[d_2, a_2]$  with the label  $v$ .

Pasting together  $\overline{P}_1$  over  $\overline{P}_2$  by their boundaries we obtain a picture  $P$  on the torus  $T$  over the presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$ . The pasted circles  $[d_1, a_1]$  and  $[a_2, d_2]$  (resp.,  $[b_1, c_1]$  and  $[c_2, b_2]$ ) form a circle  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ). The pasted points  $(ad)_1$  and  $(ad)_2$  ( $(bc)_1$  and  $(bc)_2$ ) form a point  $p_v$  ( $p_u$ ). By  $Conj$  denote a circle formed by the pasted subpaths  $Conj_1$  and  $Conj_2$ . The circles  $\underline{Equ}$  and  $\overline{Equ}$  will be called *the equators*. The points  $p_u$ ,  $p_v$  will be called *the poles*.

So  $Lab_{p_v}(\underline{Equ})$  is equal to  $v^{-1}$  or  $v$ ,  $Lab_{p_u}(\overline{Equ})$  is equal to  $u$  or  $u^{-1}$  depending on the direction of travelling along  $\underline{Equ}$  and  $\overline{Equ}$ . Fix the positive direction of travelling along the equators so that  $Lab_{p_v}^+(\underline{Equ})$  is equal to  $v$ ,  $Lab_{p_u}^+(\overline{Equ})$  is equal to  $u$ .

The equators  $\underline{Equ}$  and  $\overline{Equ}$  divide the torus  $T$  into two annulus (corresponding to  $\overline{P}_1$  and  $\overline{P}_2$ ). The annulus containing the vertices with labels from  $R_1$  (resp.,  $R_2$ ) will be called *the  $R_1$ -annulus* (resp., *the  $R_2$ -annulus*).

In the sequel we will use *admissible moves* to transform the picture  $P$  on  $T$ . A move is called *admissible* if after the move,

- (i)  $Lab_{p_v}^+(\underline{Equ})$  (resp.,  $Lab_{p_u}^+(\overline{Equ})$ ) is replaced by a word equal to  $v$  (resp.,  $u$ ) to within elements from  $N_1 \cap N_2$ ;
- (ii) all generalized vertices with labels from  $N_1$  are only in the  $R_1$ -annulus, all generalized vertices with labels from  $N_2$  are only in the  $R_2$ -annulus, where a *generalized vertex* is a vertex (one can consider it as a 'small' planar picture) with the label equal to an arbitrary (not-necessary reduced) word of  $N_1$  (or  $N_2$ );
- (iii) the equators and *Conj* remain unchanged.

**Assertion 1.** *Let the presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$  be atorical (Condition 2.2 of Theorem 1). Then there exists a finite sequence of admissible moves of  $P$ , at the end of which the labels of the equators will have one of the following form:*

- (1)  $Lab_{p_u}^+(\overline{Equ}) = \alpha(\omega'\nu_1\nu_2)\alpha^{-1}$ ,  $Lab_{p_v}^+(\underline{Equ}) = \beta(\omega'\nu_1\nu_2)\beta^{-1}$ ;
  - (2)  $Lab_{p_u}^+(\overline{Equ}) = \alpha(\omega^k\nu_1\omega^{-l}\nu_2\omega^l)\alpha^{-1}$ ,  $Lab_{p_v}^+(\underline{Equ}) = \beta(\omega^k\nu_1\nu_2)\beta^{-1}$ ;
- where  $\nu_i \in N_i$ ,  $\alpha, \beta, \omega, \omega' \in F$ ,  $l, k \in \mathbb{Z}$ ,  $l \neq 0$  can be determined by  $P$  at the end.

To prove Theorem 1 let us use Assertion 1, which will be proved in Section 3. By Assertion 1 we have two possibilities for representation of  $u$  and  $v$  to within elements from  $N_1 \cap N_2$ . If  $u$  and  $v$  have the form (1), they are evidently conjugated in  $F/N_1 \cap N_2$  by the word  $h = \alpha^{-1}\beta$ . Consider the case, when  $u$  and  $v$  have the form (2). The following notations will be used:

$$\tilde{u} = \alpha^{-1}u\alpha, \quad \tilde{v} = \beta^{-1}v\beta;$$

$$Roots_{G_1}(\tilde{v}) = \{c \in F \mid \exists s = s(c) \in \mathbb{Z} : \tilde{v} = c^s \text{ B } G_1\};$$

$$Roots_{G_2}(\tilde{v}) = \{d \in F \mid \exists t = t(d) \in \mathbb{Z} : \tilde{v} = d^t \text{ B } G_2\}.$$

**Lemma 2.** *Let the presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$  be atorical (Condition 2.2 of Theorem 1) and  $u = \alpha(\omega^k \nu_1 \omega^{-l} \nu_2 \omega^l) \alpha^{-1}$ ,  $v = \beta(\omega^k \nu_1 \nu_2) \beta^{-1}$  in  $F/N_1 \cap N_2$ . Then  $u$  and  $v$  are conjugated in  $F/N_1 \cap N_2$  if and only if there exist  $c \in \text{Roots}_{G_1}(\tilde{v})$ ,  $d \in \text{Roots}_{G_2}(\tilde{v})$ ,  $\bar{s}, \bar{t} \in \mathbb{Z}$  with  $0 \leq \bar{s} < s(c)$ ,  $0 \leq \bar{t} < t(d)$  such that  $d^{-\bar{t}} c^{-\bar{s}} \omega^l$  belongs to the cyclic subgroup  $\langle \tilde{v} \rangle$  of  $G$ .*

*Proof of Lemma 2.* Assume that there exists a word  $h \in F$  such that the equality  $u = h^{-1} v h$  holds in  $F/N_1 \cap N_2$ . Then the equality  $u = h^{-1} v h$  holds both in  $F/N_1$  and  $F/N_2$ . It is clear that  $u$  and  $v$  are conjugated by  $h$  if and only if  $\tilde{u}$  and  $\tilde{v}$  are conjugated by the word  $x$ , where  $x = \alpha^{-1} h \beta$ . Hence further we will consider  $\tilde{u}$  and  $\tilde{v}$  and investigate  $x$ .

In  $G_1 = F/N_1$ ,  $\tilde{u} = \omega^{-l}(\omega^k \nu_2) \omega^l$  and  $\tilde{v} = \omega^k \nu_2$ . Since  $\tilde{u} = x^{-1} \tilde{v} x$  in  $G_1$ , we have that  $\omega^l x^{-1}$  and  $\tilde{v}$  commute in  $G_1$ . By Condition 2.2 of Theorem 1 the presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$  is atorical, hence, the presentation  $G_1 = \langle A \mid R_1 \rangle$  is also atorical. By Theorem 13.5 [9] it follows that there exists  $c \in \text{Roots}_{G_1}(\tilde{v})$  such that  $\tilde{v} = c^s$ ,  $\omega^l x^{-1} = c^{m_1}$  in  $G_1$  for some  $s = s(c)$ ,  $m_1 \in \mathbb{Z}$ . On the other hand,  $\tilde{u}$  and  $\tilde{v}$  are equal to  $\omega^k \nu_1$  in  $G_2 = F/N_2$ . Since  $\tilde{u} = x^{-1} \tilde{v} x$  in  $G_2$ , we have that  $x^{-1}$  and  $\tilde{v}$  commute in  $G_2$ . By Condition 2.2 of Theorem 1 and Theorem 13.5 [9] there exists  $d \in \text{Roots}_{G_2}(\tilde{v})$  such that  $\tilde{v} = d^t$ ,  $x^{-1} = d^{m_2}$  in  $G_2$  for some  $t = t(d)$ ,  $m_2 \in \mathbb{Z}$ .

It follows from the equalities  $\omega^l x^{-1} = c^{m_1}$  in  $G_1$  and  $x^{-1} = d^{m_2}$  in  $G_2$  that  $\omega^l = c^{m_1} d^{-m_2}$  in  $G = F/N_1 N_2$ . Since  $\tilde{v} = c^s$  in  $G_1$ ,  $\tilde{v} = d^t$  in  $G_2$ , we have  $\omega^l = c^{\bar{s}} d^{\bar{t}} \tilde{v}^p$  in  $G$  for  $0 \leq \bar{s} < s$ ,  $0 \leq \bar{t} < t$  and some integer  $p$ , that is,  $d^{-\bar{t}} c^{-\bar{s}} \omega^l = \tilde{v}^p$  in  $G$ .

Conversely, suppose  $d^{-\bar{t}} c^{-\bar{s}} \omega^l = \tilde{v}^p$  in  $G = F/N_1 N_2$ . Let us prove that  $u$  and  $v$  are conjugated in  $F/N_1 \cap N_2$ . Since  $d^{-\bar{t}} c^{-\bar{s}} \omega^l = \tilde{v}^p$  in  $G$ , the word  $\tilde{v}^{-p} d^{-\bar{t}} c^{-\bar{s}} \omega^l$  is represented in the form  $\tilde{\nu}_2 \tilde{\nu}_1^{-1}$  for some words  $\tilde{\nu}_i \in N_i$  ( $i = 1, 2$ ). Therefore we have the equality  $c^{-\bar{s}} \omega^l \tilde{\nu}_1 = d^{\bar{t}} \tilde{v}^p \tilde{\nu}_2$  in  $F$ . Let us verify that we can take  $c^{-\bar{s}} \omega^l \tilde{\nu}_1 = d^{\bar{t}} \tilde{v}^p \tilde{\nu}_2$  as  $x$ . Indeed, in  $G_1$  we have

$$x^{-1} \tilde{v} x = x^{-1} c^s x = \omega^{-l} c^{\bar{s}} c^s c^{-\bar{s}} \omega^l = \omega^{-l} c^s \omega^l = \omega^{-l} \tilde{v} \omega^l = \omega^{-l} (\omega^k \nu_2) \omega^l = \tilde{u}.$$

In  $G_2$  we have

$$x^{-1} \tilde{v} x = x^{-1} d^t x = \tilde{v}^{-p} d^{-\bar{t}} d^t d^{\bar{t}} \tilde{v}^p = \tilde{v}^{-p} \tilde{v} \tilde{v}^p = \tilde{v} = \tilde{u}.$$

Hence,  $x^{-1} \tilde{v} x = \tilde{u}$  in  $F/N_1 \cap N_2$ . Therefore  $u = h^{-1} v h$  in  $F/N_1 \cap N_2$  for  $h = \alpha x \beta^{-1}$ . ■

By Lemma 2 we get the following algorithm.

By the word  $\tilde{v}$  determine finite sets  $\text{Roots}_{G_1}(\tilde{v})$ ,  $\text{Roots}_{G_2}(\tilde{v})$  (it is possible by Condition 1.2 of Theorem 1). For each  $c \in \text{Roots}_{G_1}(\tilde{v})$  and

$d \in \text{Roots}_{G_2}(\tilde{v})$ , using Condition 1.1 of Theorem 1, find the numbers  $s = s(c), t = t(d) \in \mathbb{Z}$  with the least absolute values such that  $\tilde{v} = c^s$  in  $G_1$  and  $\tilde{v} = d^t$  in  $G_2$ . Using Condition 2.1 of Theorem 1, verify whether there exists an integer  $p$  such that  $d^{-\bar{t}}c^{-\bar{s}}\omega^l = \tilde{v}^p$  in  $G = F/N_1N_2$  for some integers  $\bar{s}, \bar{t}$  with  $0 \leq \bar{s} < s(c), 0 \leq \bar{t} < t(d)$ . If such  $p$  is found, express  $\tilde{v}^{-p}d^{-\bar{t}}c^{-\bar{s}}\omega^l$  with defining relations  $R_1 \cup R_2$  of  $G$  (it is possible by Condition 2.1 of Theorem 1) and represent  $\tilde{v}^{-p}d^{-\bar{t}}c^{-\bar{s}}\omega^l$  in the form  $\tilde{\nu}_2\tilde{\nu}_1^{-1}$ , where  $\tilde{\nu}_i \in N_i$  ( $i = 1, 2$ ). One can take  $\alpha c^{-\bar{s}}\omega^l\tilde{\nu}_1\beta^{-1}$  as a word  $h$  conjugating  $u$  and  $v$ . If for any  $c \in \text{Roots}_{G_1}(\tilde{v})$  and  $d \in \text{Roots}_{G_2}(\tilde{v})$  there is no such  $p$ , conclude that  $u$  and  $v$  are not conjugated in  $F/N_1 \cap N_2$ . So Theorem 1 is proved. ■

## 2. ADMISSIBLE MOVES USING IN THE PROOF OF ASSERTION 1.

Below any domain  $M \subset T$  homeomorphic to the square  $\{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\}$  together with vertices and parts of edges belonging to  $M$  will be called a *map*. For a given path (an edge) on the torus  $T$ , any part of the path (the edge) homeomorphic to  $\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$  will be called a *segment* of the path (of the edge). We will say that a domain on the torus *contains nothing*, if it does not contain poles, vertices and segments of edges of  $P$ . We will say that a domain on the torus *contains absolutely nothing*, if it contains nothing and there is no point from  $\underline{Equ} \cup \overline{Equ} \cup \text{Conj}$  in it.

### 1) Isotopy.

An *isotopy* of the picture  $P$  is defined by replacing  $P$  by a picture  $F_1(P)$ , where  $F_t : T \times [0, 1] \rightarrow T \times [0, 1]$  is a continuous isotopy of the torus  $T$  such that

- (i)  $F_t$  leaves fixed all vertices and the both poles, i.e. for each  $t \in [0, 1]$  and each vertex  $V_i$ ,  $F_t(V_i) = V_i$ ,  $F_t(p_u) = p_u$ ,  $F_t(p_v) = p_v$ ;
- (ii) for each  $t \in [0, 1]$  and each edge  $E_j$  the intersection of  $F_t(E_j)$  and  $\underline{Equ}$ ,  $\overline{Equ}$ ,  $\text{Conj}$  consists of a finite number of points, moreover, if  $\underline{Equ}$ , or  $\overline{Equ}$ , or  $\text{Conj}$  intersects  $F_1(E_j)$ , then it crosses it, and doesn't just touch it.



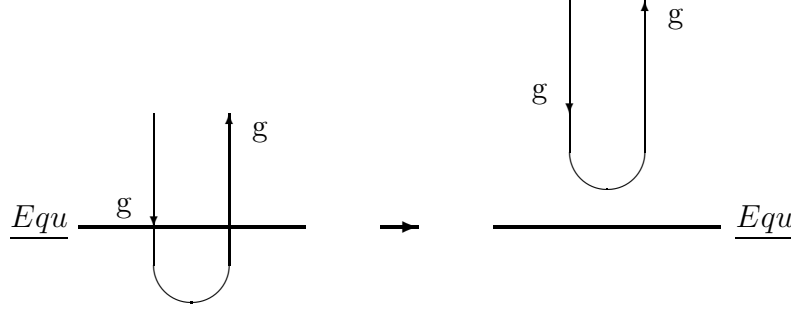


Fig. 1

An isotopy of  $P$  is an admissible move because either it corresponds to a succession of free insertions or free deletions in  $Lab_{p_v}^+(\underline{Equ})$  and  $Lab_{p_u}^+(\overline{Equ})$  or it does not change  $Lab_{p_v}^+(\underline{Equ})$  and  $Lab_{p_u}^+(\overline{Equ})$  at all (see Fig.1).

2) *Deletion of a superfluous loop (this is a particular case of isotopy).* Let  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ,  $\overline{Conj}$ ) intersect any edge  $E$  in two points, which divide  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ,  $\overline{Conj}$ ) into two parts so that one of these parts  $\zeta$  does not intersect any edge and does not contain the poles. By  $\vartheta$  denote the segment of  $E$  between these points. If a disk on the torus  $T$  encircled by the circle  $\zeta \sqcup \vartheta$  contains absolutely nothing inside, then  $\vartheta$  is called a *superfluous loop*. It is clear that superfluous loops do not contribute to the corresponding equatorial label (considered as an element of the free group). Therefore superfluous loops can be removed (see Fig.1).

3) *Bridge moves.*

Assume that a map  $M$  contains absolutely nothing except for two segments of edges  $\{x = -1/2, -1 < y < 1\}$  and  $\{x = 1/2, -1 < y < 1\}$ , which are contrariwise oriented and labelled by the same word  $g$ . A transformation of  $P$  is called a *bridge move* if it does not change  $P$  out of  $M$  and change  $P$  inside  $M$  as is shown on Fig 2. A bridge move is an admissible move because it does not change the equatorial labels.

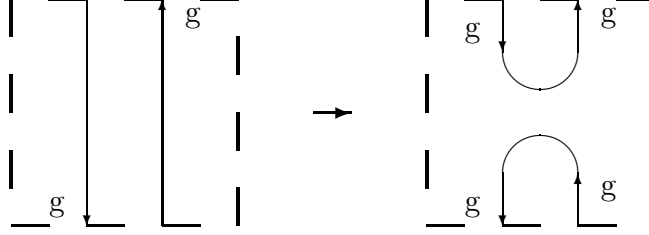


Fig. 2

4) *Uniting of edges.*

Let  $E_1$  and  $E_2$  be two edges-circles with labels  $g_1$  and  $g_2$ , which side by side intersect  $\underline{Equ}$ ,  $\overline{Equ}$  and  $Conj$  and bound on the torus an annulus, containing nothing, or  $E_1$  and  $E_2$  be two edges with labels  $g_1$  and  $g_2$ , which join the same vertices, side by side intersect  $\underline{Equ}$ ,  $\overline{Equ}$  and  $Conj$  and encircle on the torus a disk, containing nothing. Remove  $E_2$ . If  $E_1$  and  $E_2$  had the same orientation, label  $E_1$  by  $g_1g_2$  or  $g_2g_1$ , otherwise label  $E_1$  by  $g_1g_2^{-1}$  or  $g_2^{-1}g_1$ . The label for  $E_1$  should be chosen so that the contribution of this label to the equatorial labels remains the same as the contribution of the both edges  $E_1$  and  $E_2$ . We will assume that the multiplication of  $g_1$  and  $g_2^{\pm 1}$  is free.

5) *Cutting of complete dipoles.*

A *complete  $R_1$ -dipole* is a dipole  $D_1$  such that the labels of its vertices is equal to  $r_1^{\pm 1} \in R_1 \setminus R_2$  and its vertices are joined by a single edge  $E_1$  with the label  $r_1$ .

Consider a map  $M$  in the  $R_1$ -annulus such that  $M$  contains absolutely nothing except for a segment of  $E_1$ :  $\{x = 0\}$ , starting at the point  $(0, -1)$  and ending at the point  $(0, 1)$ . Cut out  $M$  from  $P$  and paste a new map  $M'$  instead of  $M$ . The new map  $M'$  contains absolutely nothing except for two vertices  $V'$ ,  $V''$  with the labels  $r_1$ ,  $r_1^{-1}$  and two edges one of which starts at  $(0, -1)$  and ends at  $V'$ , and the other one starts at  $V''$  and ends at  $(0, 1)$ . As a result one has two complete  $R_1$ -dipoles instead of one. This move is admissible because it does not change the equatorial labels.

Similarly one can define a *complete  $R_2$ -dipole* whose vertices are labelled by  $r_2^{\pm 1} \in R_2 \setminus R_1$  and a corresponding move performed in the  $R_2$ -annulus. Similarly one can define a *complete mixed dipole* whose vertices are labelled by  $r^{\pm 1} \in R_1 \cap R_2$ .

6) *Conjugation of dipoles.*

Let  $n_1 \in N_1$ . A *generalized  $N_1$ -dipole* with the label  $n_1$  is two generalized

vertices with the labels  $n_1^{\pm 1}$  and a single edge with the label  $n_1$ , joining them. For example, a complete  $R_1$ -dipole is generalized one labelled by  $r_1 \in R_1 \subset N_1$ .

Let  $D_1$  be a generalized  $N_1$ -dipole with the label  $n_1$  and  $C$  be an edge-circle with a label  $f \in F$ , encircling on  $T$  a disk, containing nothing except for  $D_1$ . In addition the edge of  $D_1$  and the edge-circle  $C$  side by side intersect  $\underline{Equ}$ ,  $\overline{Equ}$  and  $Conj$  and contribute  $(fn_1f^{-1})^{\pm 1} \in N_1$  to the labels of  $\underline{Equ}$ ,  $\overline{Equ}$  and  $Conj$ . Remove  $C$  and label the edge of  $D_1$  by  $fn_1f^{-1}$  and its generalized vertices by  $(fn_1f^{-1})^{\pm 1}$ . This move does not change the equatorial labels, hence it is admissible.

Similarly one can define a *generalized  $N_2$ -dipole* and a corresponding move of it.

7) *Deletion of a dipoles and an edge-circles not intersecting the equators.* If a generalized dipole or an edge-circle does not intersect  $\underline{Equ}$  and  $\overline{Equ}$ , then it does not contribute to  $Lab_{p_v}^+(\underline{Equ})$  and  $Lab_{p_u}^+(\overline{Equ})$ . Hence remove it.

8) *Conjugation of a pole.*

Consider the pole  $p_v$  (everything is similar for  $p_u$ ). Let  $C$  be an edge-circle with a label  $g \in F$  and  $C$  encircle on  $T$  a disk containing absolutely nothing except for  $p_v$ , only one segment of  $\underline{Equ}$  and only one segment of  $Conj$ . The union of  $C$  and  $p_v$  is called a *conjugated pole*  $p_v$ . The pole  $p_v$  itself will be considered as a conjugated pole (encircled by an edge-circle  $C$  with the label equal to the identity of the free group). If a conjugated pole  $p_v$  is surrounded in the same way by an edge-circle  $\tilde{C}$ , then unite  $C$  and  $\tilde{C}$ . This move does not change the equatorial labels, hence it is admissible.

9) *Deletion of a one-sided dipole.*

Let the edge of a generalized  $N_1$ -dipole  $D_1$  (everything is similar for a generalized  $N_2$ -dipole) with the label  $n_1 \in N_1$  do not intersect  $Conj$  and intersect only one of the equators (for definiteness,  $\overline{Equ}$ ) and only at two points. Then  $D_1$  is called a *one-sided  $N_1$ -dipole*.

There exists a closed disk  $O$  containing absolutely nothing except for  $D_1$  and two segments  $[s_1, s_2]$  and  $[t_1, t_2]$  of  $\overline{Equ}$ , where the points  $s_1, s_2, t_1, t_2$  belong to  $\partial O \cap \overline{Equ}$ . Note that the labels of  $[s_1, s_2]$  and  $[t_1, t_2]$  are equal to  $n_1$  and  $n_1^{-1}$  respectively, i.e., to the labels of  $D_1$ . In addition either  $[s_2, t_1]$  or  $[t_2, s_1]$  does not contain the pole. For definiteness let us assume that it is  $[s_2, t_1]$ . The points  $s_2, t_1$  divide  $\partial O$  into two segments. By  $\varrho$  denote such of them which contains no points of the  $R_1$ -annulus. Then the closed path  $[s_2, t_1] \cup \varrho$  encircles a planar picture over the presentation  $G = \langle A \mid R_2 \rangle$ . By Lemma 1 the label  $n_2$  of  $[s_2, t_1] \cup \varrho$  belongs to  $N_2$ . Since no edges intersect  $\varrho$ ,  $n_2$

is the label of  $[s_2, t_1]$ . So the label of  $[s_1, s_2] \cup [s_2, t_1] \cup [t_1, t_2]$  is equal to  $n_1 n_2 n_1^{-1}$ . Remove  $D_1$  from  $P$ . The label of  $[s_1, s_2] \cup [s_2, t_1] \cup [t_1, t_2]$  becomes equal to  $n_2$ . This move does not change  $Lab_{p_u}^+(\overline{Equ})$  to within  $n_1 n_2 n_1^{-1} n_2^{-1} \in N_1 \cap N_2$ . Hence this move is admissible.

10) *Permutation of two-sided dipoles.*

Let the edge of a generalized  $N_1$ -dipole  $D_1$  with the label  $n_1 \in N_1$  do not intersect  $Conj$  and intersect each of the equators  $\overline{Equ}$  and  $\underline{Equ}$  exactly at one point. Then  $D_1$  is called a *two-sided  $N_1$ -dipole*.

There is an open disk  $O_1$  containing absolutely nothing except for  $D_1$  and two segments  $[s_1, t_1] \in \overline{Equ}$  and  $[q_1, p_1] \in \underline{Equ}$ , where the points  $s_1, t_1$  belong to  $\partial O_1 \cap \overline{Equ}$ , the points  $q_1, p_1$  belong to  $\partial O_1 \cap \underline{Equ}$ . Note that the labels of  $[s_1, t_1]$  and  $[q_1, p_1]$  are equal to  $n_1$  and  $n_1^{-1}$  respectively, i.e., to the labels of  $D_1$ .

Similarly one can define a *two-sided  $N_2$ -dipole*. Substituting 2 instead of 1 in the above notations for the two-sided  $N_1$ -dipole, one gets the same notations for a two-sided  $N_2$ -dipole.

Now let both a two-sided  $N_1$ -dipole  $D_1$  and a two-sided  $N_2$ -dipole  $D_2$  be in  $P$ . The points  $s_1, s_2, t_1, t_2$  divide  $\overline{Equ}$  into four segments. Assume that one of them (say  $[t_1, s_2]$ ) does not intersect any edge and does not contain the pole. Then the label of the segment  $\sigma = [s_1, t_1] \cup [t_1, s_2] \cup [s_2, t_2]$  is equal to  $n_1 n_2$ . Permute the segments  $[s_1, t_1]$  and  $[s_2, t_2]$  (see Fig. 3).

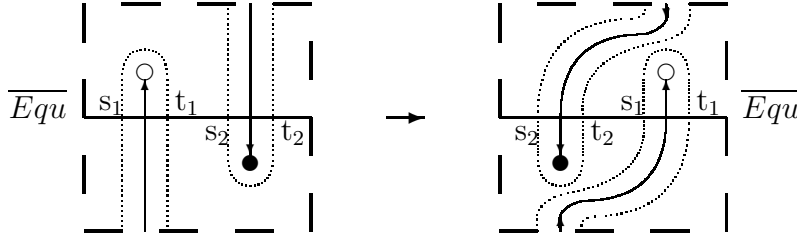


Fig. 3

After this move, the label of the new segment  $\sigma$  become equal to  $n_2 n_1$ . This move is admissible, since after it  $Lab_{p_u}^+(\overline{Equ})$  is not changed to within the word  $n_2^{-1} n_1^{-1} n_2 n_1 \in [N_1, N_2]$ .

Similarly one can define the same move for  $\underline{Equ}$ .

11) *Moving of an edge over a dipole or a pole.*

Let  $X$  be a generalized  $N_1$ - or  $N_2$ -dipole (resp., a conjugated  $p_u$  or  $p_v$  pole),  $O_1, O_2, O_3$  be three closed disks on the torus  $T$  containing nothing

except for  $X$  such that  $O_3 \subset O_2 \setminus \partial O_2$ ,  $O_2 \subset O_1 \setminus \partial O_1$ . Let  $E$  be an edge with a label  $g \in F$  such that  $E \cap O_1 = \emptyset$  and there exists a simple path  $\gamma$  joining points  $o \in E$  and  $o_3 \in \partial O_3$ , intersecting  $E$  and  $\partial O_1, \partial O_2$  exactly at one point, not intersecting other edges, the equators and  $Conj$  and not passing through any vertex. Thus  $Lab^+(\gamma) = g$ . Put in  $P$  two contrariwise oriented edge-circles  $C_1 = \partial O_1$  and  $C_2 = \partial O_2$  labelled by  $g \in F$  so that  $Lab^+(\gamma)$  becomes identically equal to  $gg^{-1}g$ . Apply the bridge move to  $E$  and  $C_1$ , the conjugation to  $X$  and  $C_2$ . It is clear that this move is admissible, because either it corresponds to an insertion of inverse words in  $Lab_{p_v}^+(\underline{Equ})$  and  $Lab_{p_u}^+(\overline{Equ})$ , or it does not change  $Lab_{p_v}^+(\underline{Equ})$  and  $Lab_{p_u}^+(\overline{Equ})$  at all.

### 3. PROOF OF ASSERTION 1.

#### **STEP 1.** *Extraction of complete dipoles.*

Since the presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$  is atorical, there exists a dipole  $D$  in the picture  $P$  on the torus  $T$ , i.e., there exists a couple of vertices  $V_1$  and  $V_2$  with mutually inverse labels  $r$  and  $r^{-1}$  such that  $V_1$  and  $V_2$  are connected by an edge  $\rho$ . Applying the bridge moves no more than  $|r| - 1$  times, we obtain that all edges go from  $V_1$  to  $V_2$  side by side in a parallel way to  $\rho$  and  $\rho$  remains unchanged. Unite these edges. This makes the dipole  $D$  complete.

Now the picture  $P$  consists of two disjoint subpictures  $P_1 \sqcup P_2$ , one of which (say  $P_1$ ) contains nothing except for the complete dipole  $D$ . The subpicture  $P_2$  is a picture on the torus over presentation  $G = \langle A \mid R_1 \cup R_2 \rangle$ . Besides  $P_2$  contains two fewer vertices than  $P$ . Repeating the above procedure for  $P_2$ , and so on, we will eventually reduce  $P$  to  $m_V/2$  complete dipoles and edge-circles, where  $m_V$  is the number of vertices in  $P$ .

#### **STEP 2.** *A move after which dipoles do not intersect $Conj$ .*

After Step 1 the picture  $P$  consists of edges-circles, complete  $R_1$ -,  $R_2$ -dipoles and complete mixed dipoles. If the edges of some complete dipoles do not intersect the equators, remove these complete dipoles. Also remove complete mixed dipoles from  $P$ . This changes the equatorial labels by elements from  $R_1 \cap R_2 \subset N_1 \cap N_2$ . Now  $P$  contains only complete  $R_1$ -,  $R_2$ -dipoles and edges-circles.

**Operation 1.** Consider a complete  $R_1$ -dipole  $D$  (the case of a complete  $R_2$ -dipole is similar). Let its edge intersect  $Conj_1$  at points  $o_1, \dots, o_{\tilde{m}}$ . Near by  $o_i$  ( $i = 1, \dots, \tilde{m}$ ) cut  $D$  into three complete dipoles  $D_1, D_2, D_3$ , one of which ( $D_2$ ) lies in the  $R_1$ -annulus as a whole and its edge intersects  $Conj_1$  exactly at one point (at  $o_i$ ). Remove  $D_2$  from  $P$ . Repeating the same procedure to each of  $\tilde{m}$  intersections, instead of one

complete  $R_1$ -dipole  $D$ , we obtain  $\tilde{m} + 1$  complete  $R_1$ -dipoles, neither of which intersects  $Conj_1$ .

Apply Operation 1 to each of complete  $R_1$ - and  $R_2$ -dipoles. This gives that the edges of the complete  $R_1$ -dipoles do not intersect  $Conj_1$  and the edges of the complete  $R_2$ -dipoles do not intersect  $Conj_2$ .

**Operation 2.** Consider  $Conj_1$  (the case of  $Conj_2$  is similar). It can be intersected only by the edges of complete  $R_2$ -dipoles and by edges-circles. Let  $\rho_1, \dots, \rho_{\tilde{m}}$  be edges-circles not conjugating the poles and edges of complete  $R_2$ -dipoles such that  $\rho_1, \dots, \rho_{\tilde{m}}$  intersect  $Conj_1$ , and we encounter them in the order  $\rho_1, \dots, \rho_{\tilde{m}}$  if we start at the conjugated pole  $p_u$  and travel along  $Conj_1$  to the conjugated pole  $p_v$ . Starting with  $\rho_1$ , move consecutively each edge  $\rho_i$  over the conjugated pole  $p_u$ . This gives that the edges of the complete  $R_2$ -dipoles intersect only  $Conj_2$ . Apply Operation 1 to these complete  $R_2$ -dipoles.

After Operation 2 applying to  $Conj_1$  and  $Conj_2$ , the picture  $P$  consists of edges-circles and only of complete  $R_1$ - and  $R_2$ -dipoles  $D_1, \dots, D_{\tilde{m}}$  not intersecting  $Conj$ . For each  $D_i$ , by  $m_i$  denote the number of intersections of the equators and the edge of  $D_i$ . Note that  $m_i$  is even. One can assume that  $m_i > 0$ , otherwise remove  $D_i$  from  $P$ . If  $m_i > 2$ , cut  $D_i$  into  $m_i/2$  complete dipoles each of which intersects the equators exactly at two points. This move applying to each  $D_i$  makes all dipoles either one-sided or two-sided. Remove all one-sided dipoles from  $P$ .

**STEP 3.** *Getting rid of contractible edges-circles.*

After Step 2 the picture  $P$  consists just of two-sided dipoles and edges-circles. Call an edge-circle *contractible*, if it divides the torus into two parts one of which is homeomorphic to a disk. This part will be called the *interior* of the edge-circle.

After the deletions of superfluous loops from the equators and  $Conj$  each contractible edge-circle  $C$  belongs to one of the following types.

- I) The interior of  $C$  contains absolutely nothing.
- II) The interior of  $C$  contains nothing except for just one two-sided dipole or just one conjugated pole.
- III) In the interior of  $C$ , there are at least two two-sided dipoles, or at least one two-sided dipole and at least one conjugated pole, or the both conjugated poles.

Remove all edges-circles of Type I from  $P$ . Apply the conjugation of dipoles or the conjugation of poles to all edges-circles of Type II. Now just  $\tilde{m}$  edges-circles of Type III remain in  $P$ .

Call an edge-circle of Type III *minimal*, if there is no other edges-circles of Type III in its interior.

**Operation 3.** Let  $C$  be a minimal edge-circle of Type III with  $\check{m}_1$  two-sided dipoles and  $\check{m}_2$  conjugated poles in its interior,  $\check{m}_1 + \check{m}_2 \geq 2$ . It is clear that by the isotopy and the  $\check{m}_1 + \check{m}_2 - 1$  bridge moves,  $C$  can be reduced to  $\check{m}_1 + \check{m}_2$  edges-circles of Type II. Apply the conjugation to each of these edges-circles of Type II.

Operation 3 gives a picture  $P$  with one fewer edges-circles. Hence after no more than  $\check{m}$  applications of Operation 3, the picture  $P$  will contain just non-contractible edge-circles and two-sided dipoles.

**STEP 4.** *Uniting of non-contractible edges-circles.*

After Step 3 the picture  $P$  contains just non-contractible edges-circles  $Z_1, \dots, Z_m$  and two-sided dipoles. Cutting out one of the edges-circles ( $Z_1$ ) from the torus converts the torus to a surface  $\Omega$  homeomorphic to an annulus. In  $\Omega$  any closed simple not contractible path ( $Z_i, i \neq 1$ ) is homotopic to the boundary ( $Z_1$ ) and to any other closed simple not contractible path ( $Z_j, j \neq 1, i$ ) disjoint with it. The edges-circles  $Z_2, \dots, Z_m$  divide  $\Omega$  into  $m$  disjoint parts  $\Omega_1, \dots, \Omega_m$  each homeomorphic to an annulus. Assume that the edges-circles  $Z_2, \dots, Z_m$  are numbered so that  $\Omega_1$  are bounded by  $Z_1$  and  $Z_2$ ,  $\Omega_2$  are bounded by  $Z_2$  and  $Z_3, \dots, \Omega_m$  are bounded by  $Z_m$  and  $Z_1$ .

Consider  $\Omega_1$ . If there are conjugated poles or two-sided dipoles in  $\Omega_1$ , apply the isotopy and move  $Z_2$  over these dipoles and poles to transpose these poles and dipoles from  $\Omega_1$  to  $\Omega_2$ , and to approach  $Z_2$  and  $Z_1$  to each other so that  $Z_1$  and  $Z_2$  become parallel and side by side intersect the equator and *Conj*. Repeat the same procedure for each  $\Omega_i, i = 2, \dots, m - 1$  to transpose conjugated poles and generalized dipoles from  $\Omega_i$  to  $\Omega_{i+1}$ . We will eventually obtain that all conjugated poles and two-sided dipoles of  $P$  are in  $\Omega_m$  and  $Z_1, \dots, Z_m$  are parallel and side by side intersect the equators and *Conj*. Unite  $Z_1, \dots, Z_m$ . This gives a single edge-circle  $Z$ .

**STEP 5.** *Disposition of two-sided dipoles in the order.*

Above the orientation on the equators was fixed. If we start at  $p_v$  and travel once around  $\underline{Equ}$  in the positive direction, we encounter a succession of edges of dipoles  $D_1, \dots, D_s$  intersecting  $\underline{Equ}$ . We say that an  $N_2$ -dipole  $D_i$  and an  $N_1$ -dipole  $D_j$  form the *inversion* on  $\underline{Equ}$ , if  $i < j$ , otherwise they form the *order* on  $\underline{Equ}$ . In the same way one can define the inversion and the order on  $\overline{Equ}$ . We will say that one circuit along  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ) in the positive direction starting at  $p_v$  (resp.,  $p_u$ ) is a movement *from the left to the right*.

The edge-circle  $Z$  is divided by the equators into segments. *Two-sided  $R_2$ -pieces* (resp., *two-sided  $R_1$ -pieces*) are such of these segments

that do not intersect  $Conj$  and lie in the  $R_2$ -annulus (resp., in the  $R_1$ -annulus) at the whole, starting on one of the equators and ending on the other one.

**Lemma 3.** *There exists a finite succession of admissible moves that disposes all edges of two-sided  $N_1$ -dipoles in the  $R_2$ -annulus on the left side of the two-sided  $R_2$ -pieces of  $Z$ .*

*Proof of Lemma 3.* If there are no two-sided  $N_1$ -dipole or two-sided  $R_2$ -pieces in  $P$ , there is nothing to prove. Otherwise let  $m$  be the minimal number of transpositions to get all edges of two-sided  $N_1$ -dipoles on the left side of the two-sided  $R_2$ -pieces of  $Z$ . If  $m = 0$ , there is nothing to prove. Otherwise consider the rightmost two-sided  $N_1$ -dipole  $D$  which has a two-sided  $R_2$ -piece  $\rho$  on the left such that there are no other  $N_1$ -dipoles or two-sided  $R_2$ -pieces between  $D$  and  $\rho$ . Move  $\rho$  over  $D$  to the right of  $D$ . This decreases  $m$  by 1. Now use induction on  $m$ . ■

The edges of two-sided  $N_1$ -dipoles consecutively intersect  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ). For a given two-sided  $N_1$ -dipole, let  $o'$  and  $o''$  be two consecutive intersections of its edge and  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ). By Lemma 3, removing superfluous loops, if necessary, either there are no intersections with  $Z$  between  $o'$  and  $o''$ , or there are intersections with the edges of two-sided  $N_2$ -dipoles between  $o'$  and  $o''$  and  $Z$  intersects  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ) between  $o'$  and  $o''$ , gets into the  $R_2$ -annulus, envelops a vertex of at least one of these two-sided  $N_2$ -dipoles, turns back to  $\underline{Equ}$  (resp.,  $\overline{Equ}$ ) and returns to the  $R_1$ -annulus.

**Lemma 4.** *There exists a finite succession of admissible moves that disposes all two-sided dipoles of  $P$  in the order.*

*Proof of Lemma 4.* By  $m'$  denote the number of inversions on  $\underline{Equ}$ , by  $m''$  the number of inversions on  $\overline{Equ}$ . If  $m' + m'' = 0$ , there is nothing to prove. Let  $m' + m'' > 0$ .

If  $m' > 0$ , at first consider  $\underline{Equ}$ . Let  $D_1$  and  $D_2$  be two neighboring two-sided  $N_1$ - and  $N_2$ -dipoles forming the inversion on  $\underline{Equ}$  such that there are no other dipoles between them. We can assume that  $D_2$  is not enveloped by  $Z$ , otherwise move  $Z$  over  $D_2$ . The permutation of  $D_1$  and  $D_2$  decreases  $m'$  by 1. Induction on  $m'$  gives that all two-sided dipoles form the order on  $\underline{Equ}$ .

If  $m'' > 0$ , apply the same procedure to  $\overline{Equ}$ . ■

**Lemma 5.** *There exists a finite succession of admissible moves that disposes all edges of two-sided  $N_2$ -dipoles in the  $R_2$ -annulus on the left side of two-sided  $R_1$ -pieces of  $Z$ .*



The proof of Lemma 5 is similar to the proof of Lemma 3.

So all two-sided dipoles of  $P$  form the order both on  $\overline{Equ}$  and on  $\underline{Equ}$ . In addition two-sided  $N_1$ -dipoles (resp.,  $N_2$ -dipoles) are near by to each other and intersect the equators side by side. Replace all these two-sided  $N_1$ -dipoles (resp.,  $N_2$ -dipoles) by one two-sided dipole  $\Delta_1$  (resp.,  $\Delta_2$ ) with the edge's label  $\nu_1$  (resp.,  $\nu_2$ ) equal to the product of the labels of all these two-sided  $N_1$ -dipoles (resp.,  $N_2$ -dipoles), i.e.,  $\nu_1$  (resp.,  $\nu_2$ ) belongs to  $N_1$  (resp.,  $N_2$ ). If  $\nu_1$  (resp.,  $\nu_2$ ) is equal to the identity in  $F$ , remove the dipole  $\Delta_1$  (resp.,  $\Delta_2$ ).

**STEP 6. *Finale.***

The picture  $P$  can contain at most one edge-circle  $Z$ , at most one two-sided  $N_1$ -dipole  $\Delta_1$  (with the label  $\nu_1$ ), at most one two-sided  $N_2$ -dipole  $\Delta_2$  (with the label  $\nu_2$ ) and two conjugated poles  $p_u$  and  $p_v$ . By  $\alpha$  (resp.,  $\beta$ ) denote the label of the edge conjugating the pole  $p_u$  (resp.,  $p_v$ ). Below  $P$  will be transformed by isotopy, by moving  $Z$  over  $\Delta_1$  and  $\Delta_2$ , by conjugation of poles. For simplicity of notation the labels of  $\Delta_1$  and  $\Delta_2$ , the labels of edges conjugating  $p_u$  and  $p_v$  will be again denoted by  $\nu_1, \nu_2, \alpha, \beta$ .

There are three possibility:

**Case A.** *There is no  $Z$  in  $P$ .*

We have Case (1) of Assertion 1, i.e.  $Lab_{p_u}^+(\overline{Equ}) = \alpha(\nu_1\nu_2)\alpha^{-1}$ ,  $Lab_{p_v}^+(\underline{Equ}) = \beta(\nu_1\nu_2)\beta^{-1}$ .

**Case B.** *There is  $Z$  in  $P$  and  $Z$  is homotopic to  $Conj$ .*

By isotopy, moving  $Z$  over  $\Delta_1, \Delta_2$  and the conjugated poles, dispose  $Z$  near by  $Conj$  in a parallel way to  $Conj$  so that  $Z$  intersects each of the equators exactly at one point. Thus we have Case (1) of Assertion 1, i.e.,  $Lab_{p_u}^+(\overline{Equ}) = \alpha(\omega'\nu_1\nu_2)\alpha^{-1}$ ,  $Lab_{p_v}^+(\underline{Equ}) = \beta(\omega'\nu_1\nu_2)\beta^{-1}$ , where  $\omega'$  is the label of  $Z$ .

**Case C.** *There is the edge-circle  $Z$  in  $P$  and  $Z$  is homotopic to a simple closed path circuiting  $Conj$   $|k|$  times and the equators  $|l|$  times, where  $l, k \in \mathbb{Z}$ ,  $|l| \geq 1$ .*

If there is no dipole  $\Delta_2$  in  $P$ , by isotopy and moving  $Z$  over  $\Delta_1$  and the conjugated poles, dispose  $Z$  in such a way that the  $|k|$  circuits of  $Z$  along  $Conj$  are near by  $Conj$  and the  $|l|$  circuits of  $Z$  along the equators are in the  $R_1$ -annulus. Thus we have Case (1) of Assertion 1, i.e.,  $Lab_{p_u}^+(\overline{Equ}) = \alpha(\omega^k\nu_1)\alpha^{-1}$ ,  $Lab_{p_v}^+(\underline{Equ}) = \beta(\omega^k\nu_1)\beta^{-1}$ , where  $\omega$  is the label of  $Z$ .

It remains to consider the case when there exists  $\Delta_2$  in  $P$ . By isotopy and moving  $Z$  over  $\Delta_1, \Delta_2$  and the conjugated poles, dispose  $Z$  in such a way that the  $|k|$  circuits of  $Z$  along  $Conj$  are near by  $Conj$  and the  $|l|$  circuits of  $Z$  along the equators start in the  $R_1$ -annulus, go in a

parallel way to each other to the edge of  $\Delta_2$ , envelope its vertex after intersecting  $\overline{Equ}$  and return to the  $R_1$ -annulus. Thus we have Case (2) of Assertion 1:  $Lab_{p_u}^+(\overline{Equ}) = \alpha(\omega^k \nu_1 \omega^{-l} \nu_2 \omega^l) \alpha^{-1}$ ,  $Lab_{p_v}^+(\overline{Equ}) = \beta(\omega^k \nu_1 \nu_2) \beta^{-1}$ , where  $\omega$  is the label of  $Z$ . ■

**Remark 1.** *It follows from the proof of Assertion 1 that the integer  $L = L(u, v, R_1, R_2)$  such that  $|\alpha|, |\beta|, |\omega|, |l| \leq L$ , can be chosen as  $90(|h_1| + |h_2|)(1 + 2l_R + 2l_R^2 + \dots + 2l_R^{m_V/2-1})$ , where  $l_R$  is the length of the longest word of  $R_1 \cup R_2$ ,  $m_V$  is the number of vertices in the initial picture  $P$ .*

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